

# Approximate Analytical Solution to Nonlinear Buckling of Graphene Beam

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**ABSTRACT:** In this paper, buckling of an Euler-Bernoulli beam made of graphene under compressive axial load is considered. The problem is formulated as nonlinear two-point boundary value problem, and the method of averaging is used to explore the effects of the nonlinearity of the graphene material on the amplitude of deflection and buckling load at different mode shapes of the beam. The results of this paper agree with results from other different analytical and numerical methods. However, the current approach has no convergence problems at high nonlinearity and solves for deflection at any mode shape giving an important advantage over the other numerical methods used.

**KEYWORDS:** Graphene, Euler-Bernoulli beam, buckling, method of averaging.

## I. INTRODUCTION

Graphene sheets in the free state show interesting properties and have promising applications in many engineering fields [1-9]. Nanoindentation of monolayer graphene membrane showed unique mechanical properties [10]. Observations of force deformation necessitate a nonlinear stress strain relation with Young's modulus  $E = 340 \pm 40 \text{ N/m}$  and nonlinear third order elastic stiffness  $D = -690 \pm 120 \text{ N/m}$  [11,12]. As a result of these properties different behavior is expected from graphene products when subjected to various loading conditions. The use of graphene in nanodevices necessitates further quantitative investigations beyond primary treatment or curve fitting of provided experimental data.

Buckling behavior of an Euler-Bernoulli beam made of graphene examined in [13-15] utilized the nonlinear mathematical models of the stress strain relation of graphene based on experimental data. In a previous research a numerical scheme is proposed to solve for the deflection of the beam at buckling and the corresponding buckling load. However, this approach allowed solving only for first mode shape of the beam. In this research an analytical technique based on the averaging method will be used to solve for the beam deflection and buckling loads at any mode shape. The nonlinearity of graphene material manifests itself through a nonlinear parameter appearing in the equilibrium equations. And one advantage of the current approach is that this nonlinear parameter doesn't have to be small when applying the averaging method.

In this paper the eigenvalue problem related to buckling of a graphene beam subject to axial compressive force will be solved. The method of averaging will be used to calculate the deflection and the buckling loads simultaneously at each mode shape of the beam. In the second section, the problem of buckling of a simply supported beam made of graphene will be reviewed. In the third section, the effect of nonlinearity on the amplitude and phase of deflection will be investigated, assuming that the amplitude and phase are slowly varying along the beam's longitudinal axis. In the fourth section we discuss the results and in the fifth section we give some concluding remarks.

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## II. SIMPLY SUPPORTED ELASTIC BEAM

Fig. 1 depicts a simply supported elastic beam of length  $L$  Subject to an axial compressive load  $P$ . If  $EI$  is the flexural rigidity of the beam's material, the equilibrium equation for the beam reads

$$EIv'''' + Pv'' = 0, 0 < x < L \quad (1.a)$$

where  $v$  is the lateral deflection and  $x$  is the distance measured along the longitudinal axis of the beam. At the end points **A** and **B** both the deflection and the bending moment vanish, i.e.

$$v(0) = v''(0), v(L) = v''(L) = 0 \quad (1.b)$$

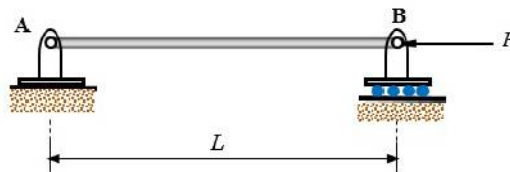


Fig. 1. Simply Supported Beam Subject to Axial Compressive Load

Integrating (1.a) twice and using the boundary conditions (1.b) one obtains

$$v'' + \lambda_0 v = 0, \lambda_0 = \frac{P}{EI} \quad (2)$$

The general solution of (2) takes the form

$$v(x) = A \cos(\sqrt{\lambda_0} x + \phi_0), \quad (3)$$

where  $A$  and  $\phi_0$  are constants. Using again the boundary conditions in (1.b) the  $n^{th}$  solutions of the deflection  $v_n(x)$  and its corresponding buckling load  $P_n$ , take the form

$$v_n(x) = \sin\left(\frac{n\pi}{L} x\right), P_n = \left(\frac{n\pi}{L}\right)^2 EI, n = 1, 2, 3, \dots, \quad (4)$$

where  $n = 1$  gives the first nontrivial solution known as the first buckling mode and its corresponding load known as the critical buckling load  $P_{crit} = (\pi/L)^2 EI$ .

## III. BUCKLING OF GRAPHENE BEAM

In the previous section the stress-strain relation used is the Hook'e law  $\sigma = E\varepsilon_x$  and the cross section of the beam is assumed to be perpendicular to its axis. But, in case of graphene, when observing deflection due to applied force the atomic force microscope (AFM) indicated that graphene may obey a nonlinear stress strain relation [8,9,12].

$$\sigma = E\varepsilon_x + D|\varepsilon_x|\varepsilon_x, \quad (5.a)$$

where

$$D = -\frac{E^2}{4\sigma_{max}}, \quad (5.b)$$

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is a negative nonlinear elastic stiffness and  $\sigma_{\max}$  is the maximum or failure stress for graphene obtained from stress-strain curve when  $\frac{d\sigma}{d\varepsilon} = 0$ . The strain at this point is given by  $\varepsilon = -\frac{E}{2D}$ .

Using (5.a) and introducing new dimensionless variables  $z = \frac{x}{L}$ ,  $u = \frac{v}{L}$ , the equilibrium equation of the graphene beam can be modeled by [11,16]

$$u'''' - \alpha(|u''|u'') + \lambda u = 0, \tag{6.a}$$

where  $\alpha = \frac{PL^2}{EI_2}$ ,  $\lambda = \frac{|D|I_3}{EI_2L}$  and  $I_2, I_3$  are the second and third moments of the beam's cross-sectional area respectively, with the boundary conditions:

$$u(0) = u''(0) = 0, u(1) = u''(1) = 0, \tag{6.a}$$

Integrating (6.a) twice and using the boundary conditions in (6.b) one obtains

$$u'' - \alpha(|u''|u'') + \lambda u = 0, \tag{7.a}$$

$$u(0) = u(1) = 0, \tag{7.b}$$

Neglecting the nonlinear term in (7.a) the solution of the linear equation in (3) can be rewritten as

$$u(x) = A \sin(\psi), \psi = (\omega x + \phi), \tag{8.a}$$

where

$$u'(x) = A\omega \cos(\psi), \tag{8.b}$$

$$u''(x) = -A\omega^2 \sin(\psi) \tag{8.c}$$

and  $\omega = \sqrt{\lambda}$ . Assuming that  $A$  and  $\phi$  are functions of  $x$ , then differentiating (8.a) twice with respect to  $x$  and equating  $u'(x)$ ,  $u''(x)$  with their values in (8.b), (7.a) we obtain

$$A' \sin(\psi) + A\phi' \cos(\psi) = 0, \tag{9.a}$$

$$A' \cos(\psi) - A\phi' \sin(\psi) = \frac{\alpha}{\omega} (|u''|u''), \tag{9.b}$$

Solving (9.a), (9.b) for  $A'$  and  $\phi'$  gives

$$A' = \frac{\alpha}{\omega} (|u''|u'') \cos(\psi) \tag{10.a}$$

$$\phi' = -\frac{\alpha}{A\omega} (|u''|u'') \sin(\psi) \tag{10.b}$$

The fact that the right hand sides of (10.a), (10.b) are periodic in the angular argument  $\psi$  and assuming that both  $A, \phi$  do not drastically change along the beam's axis we can average (10.a) and (10.b), over one period of  $\psi$ , then integrate giving

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$$A' = -\frac{\alpha A^2 \omega^3}{2\pi} \int_0^{2\pi} |\sin(\psi)| \sin(\psi) \cos(\psi) d\psi = 0 \quad (11.a)$$

$$\phi' = \frac{\alpha A^2 \omega^3}{2\pi} \int_0^{2\pi} |\sin(\psi)| \sin^2(\psi) d\psi = \frac{4\alpha A \omega^3}{3\pi} \quad (11.b)$$

### IV. RESULTS AND FINDINGS

Equation (11.a) indicates that the nonlinearity doesn't affect the amplitude of oscillations along the beam's axis while the phase angle changes linearly along the beam's axis. Substituting (11.b) into  $\phi = \phi' x + \phi_0$  gives

$$\phi = \frac{4\alpha A \omega^3}{3\pi} x + \phi_0, \quad (12)$$

and (8.a) becomes

$$u(x) = A \sin\left(\omega\left(1 + \frac{4\alpha A \omega^2}{3\pi}\right)x + \phi_0\right) \quad (13)$$

using the boundary conditions (7.b) gives

$$\phi_0 = 0, \quad (14.a)$$

$$\frac{4\alpha A}{3\pi} \omega^3 + \omega - n\pi = 0, \quad (14.b)$$

where  $n = 1, 2, 3, \dots$  defines the mode shapes for the beam's deformation. Equation (14.b) can be solved for  $\omega$  which is related to the critical buckling load corresponds to each  $n$ . One feasible solution to (14.b) is to express  $\omega$  as series expansion in  $\alpha$  giving

$$\lambda = \omega^2 = n^2 \pi^2 \left( 1 - \frac{8(n^2 \pi A)}{3} \alpha + \frac{112(n^2 \pi A)^2}{9} \alpha^2 - \frac{640(n^2 \pi A)^3}{9} \alpha^3 + \dots \right) \quad (15)$$

For  $\alpha$  sufficiently small a two-terms approximation of (15) gives the same results as in [14,15]. This approximation assumed a linear relation between  $\lambda$  and  $\alpha$  in which  $\lambda$  monotonically decreases with  $\alpha$ . However, noticing that (14.b) has one real root we can, numerically, obtain an exact value of  $\omega$  and  $\lambda$ . Fig.2 depicts the eigen value  $\lambda$  with  $\alpha$  at different mode shapes  $n$ . It is obvious that at certain mode shape the buckling load changes nonlinearly with  $\alpha$  but still monotonically decreases. For small values of  $\alpha$  the curve can be approximated as a straight line with large negative slope in  $\lambda$  with  $\alpha$ . And at higher values of  $\alpha$  the curve can be approximated as a straight line with small negative slope asymptotically approaching a horizontal line as  $\lambda \rightarrow 0$ . At a fixed value of  $\alpha$  the higher mode shapes require larger load. An interesting effect of  $\alpha$  to be noted is that at certain load a higher mode shape may be obtained by increasing  $\alpha$ . This behavior can be attributed to the negative elastic modulus  $D$  in (5) and the system may be known as hyperelastic softening.

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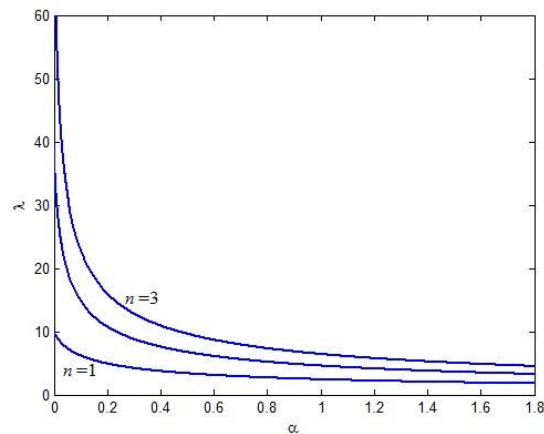


Fig. 2. Buckling Load  $\lambda$  vs  $\alpha$  for Shapes  $n = 1, 2, 3$

### V. CONCLUSION

An exact approximate analytical solution to the problem of buckling of a beam made of Graphene under compressive axial load is obtained. The results showed agreement with other results obtained by using numerical and analytical approaches. However, the current approach showed advantages over the other approaches. While other approaches are valid at small nonlinearity and for the first mode shape our results valid at high nonlinearity and for any mode shape.

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